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On the stochastic pendulum with Ornstein–Uhlenbeck noise

Kirone Mallick[1,2] **and Philippe Marcq**[1,2]

[1] Service de Physique Théorique, Centre d'Études de Saclay, 91191 Gif-sur-Yvette Cedex, France
[2] Institut de Recherche sur les Phénomènes Hors Équilibre, Université de Provence, 49 rue Joliot-Curie, BP 146, 13384 Marseille Cedex 13, France

E-mail: mallick@spht.saclay.cea.fr and marcq@irphe.univ-mrs.fr

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Abstract

We study a frictionless pendulum subject to multiplicative random noise. Because of destructive interference between the angular displacement of the system and the noise term, the energy fluctuations are reduced when the noise has a non-zero correlation time. We derive the long time behaviour of the pendulum in the case of Ornstein–Uhlenbeck noise by a recursive adiabatic elimination procedure. An analytical expression for the asymptotic probability distribution function of the energy is obtained and the results agree with numerical simulations. Lastly, we compare our method with other approximation schemes.

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1. Introduction

The behaviour of a nonlinear dynamical system is strongly modified when randomness is taken into account: noise can shift bifurcation thresholds, create new phases (noise-induced transitions), or even generate spatial patterns [1–5]. The interplay of noise with nonlinearity gives rise to a variety of phenomena that constantly motivate new research of theoretical and practical significance. Stochastic resonance [6] and biomolecular Brownian motors [7, 8] are celebrated examples of nonlinear random systems of current interest. In particular, stochastic ratchets have generated a renewed interest in the study of simple mechanical systems subject to random interactions, the common ancestor of such models being Langevin's description of Brownian motion. Many unexpected phenomena appear when one generalizes Langevin's equations to include, e.g., inertial terms, nonlinearities, external (multiplicative) noise or noise with finite correlation time; each of these new features opens a field of investigations that calls for specific techniques or approximation schemes [9]. Nonlinear oscillators with parametric noise are often used as paradigms for the study of these various effects and related mathematical methods [10]. The advantage of such models is that they have an appealing physical interpretation and appear as building blocks in many different fields; they can be simulated on a computer or constructed as real electronic or mechanical systems [11, 12]. Moreover, the mathematical apparatus needed to analyse them remains relatively elementary (as compared to the perturbative field-theoretical methods required for spatio-temporal systems [13]) and can be expected to yield exact and rigorous results.

In the present work, we study the motion of a frictionless pendulum with parametric noise, which can be physically interpreted as a randomly vibrating suspension axis. We show that the long time behaviour of a stochastic pendulum driven by a coloured noise with finite correlation time is drastically different from that of a pendulum subject to white noise. Whereas the average energy of the white-noise pendulum is a linear function of time, that of the coloured-noise pendulum grows only as the square-root of time. Our analysis is based on a generalization of the averaging technique that we have used previously for nonlinear oscillators subject to white noise: in [14–16] an effective dynamics for the action variable of the system is derived after integrating out the fast angular variable, and is then exactly solved. However, this averaging technique as such ceases to apply to systems with coloured noise when the time scale of the fast variable becomes smaller than the correlation time of the noise. Correlations between the fast variable and the noise modify the long time scaling behaviour of the system and therefore must be taken into account. We shall develop here a method that systematically retains these correlation terms before the fast variable is averaged out. This will allow us to derive analytical expressions for the asymptotic probability distribution function (PDF) of the energy of the stochastic pendulum, and to deduce the long time behaviour of the system. Our analytical results are verified by numerical simulations. Finally, we shall compare our method and results with some known approximation schemes used for multivariate systems with coloured noise. We shall show in particular that small correlation time expansions cannot explain the anomalous diffusion exponent when truncated at any finite order. The partial summation of Fokker–Planck-type terms, used to derive a 'best effective Fokker–Planck equation' for coloured noise [17, 18], leads to results that agree with ours. We emphasize that the noise considered in this work has a finite correlation time and its auto-correlation function does not have long time tails.

This paper is organized as follows. In section 2, we analyse the case where the parametric fluctuations of the pendulum are modelled by the Gaussian white noise, and explain heuristically why the coloured noise leads to an anomalous scaling of the energy. In section 3, we study the case of Ornstein–Uhlenbeck noise and explain how a recursive adiabatic elimination of the fast variable can be performed. This allows us to derive analytical results for the PDF of the energy, which we validate with direct numerical simulations. In section 4, we compare our results with effective Fokker–Planck approaches. The concluding remarks are presented in section 5.

2. The pendulum with parametric noise

The dynamics of a non-dissipative classical pendulum with parametric noise can be described by the following system of stochastic differential equations,

$$\dot{\Omega} = -(\omega^2 + \xi(t)) \sin\theta \tag{1}$$

$$\dot{\theta} = \Omega \tag{2}$$

where θ represents the angular displacement and Ω the angular velocity. The energy E of the system is

$$E = \frac{\Omega^2}{2} - \omega^2 \cos\theta. \tag{3}$$

Equations (1), (2) describe the motion of a pendulum of frequency ω whose suspension point is subject to a stochastic force proportional to the random function $\xi(t)$. Our aim is to study how the stochastic properties of $\xi(t)$ are transferred to the dynamical variables (θ, Ω) through a multiplicative and nonlinear coupling. We are chiefly interested in the case where $\xi(t)$ has a non-zero correlation time, but we first consider the Gaussian white noise. Equations (1), (2) as well as all the stochastic differential equations below are interpreted according to the rules of Stratonovich calculus.

2.1. The white noise case

The dynamics of an oscillator subject to multiplicative white noise has been studied by a number of investigators [11, 19–21]. In particular, Lindenberg *et al* have shown [22] that the physical origin of the energetic instability of a stochastic oscillator can be quantitatively related to the parametric resonance of the underlying deterministic oscillator. In this section, an instability of the same type is found for the stochastic pendulum subject to parametric white noise, using the adiabatic averaging method.

Let $\xi(t)$ be a Gaussian white noise of zero mean value and amplitude $\mathcal{D}$:

$$\langle \xi(t) \rangle = 0 \qquad \langle \xi(t)\xi(t') \rangle = \mathcal{D}\delta(t - t'). \tag{4}$$

Using the elementary dimensional analysis, we note from equation (1) that $\dot{\Omega} \sim \xi$, i.e., the angular velocity Ω grows as $t^{1/2}$ and therefore $\theta \sim t^{3/2}$. This observation can be put on a stronger basis by using the Fokker–Planck equation, associated with equations (1) and (2), that describes the evolution of the probability distribution function $P_t(\theta, \Omega)$:

$$\frac{\partial P_t}{\partial t} = -\frac{\partial}{\partial \theta}(\Omega P_t) + \frac{\partial}{\partial \Omega}(\omega^2 \sin\theta P_t) + \frac{\mathcal{D}}{2}\sin^2\theta \frac{\partial^2 P_t}{\partial \Omega^2}. \tag{5}$$

From equation (2), we observe that the angular variable θ varies rapidly as compared to Ω. Thus, following [14], we assume that, in the long time limit, the angle θ is uniformly distributed over $[0, 2\pi]$. This allows us to average equation (5) over the angular variable and to derive an effective Fokker–Planck equation for the marginal distribution $\tilde{P}_t(\Omega)$:

$$\frac{\partial \tilde{P}_t}{\partial t} = \frac{\mathcal{D}}{4}\frac{\partial^2 \tilde{P}_t}{\partial \Omega^2} \tag{6}$$

where we have replaced $\sin^2\theta$ by its mean value $1/2$. From this effective Fokker–Planck equation, we readily deduce that Ω is a Gaussian variable with PDF

$$\tilde{P}_t(\Omega) = \frac{1}{\sqrt{\pi \mathcal{D} t}} \exp\left(-\frac{\Omega^2}{\mathcal{D} t}\right). \tag{7}$$

We thus recover that Ω grows as $t^{1/2}$ when $t \to \infty$. Because $E \simeq \Omega^2/2$ (up to a term that remains bounded), we deduce the PDF of the energy

$$\tilde{P}_t(E) = \sqrt{\frac{2}{\pi \mathcal{D} t}} E^{-\frac{1}{2}} \exp\left(-\frac{2E}{\mathcal{D} t}\right). \tag{8}$$

From equation (8), we obtain the scaling behaviour of the average energy

$$\langle E \rangle = \frac{\mathcal{D}}{4} t \tag{9}$$

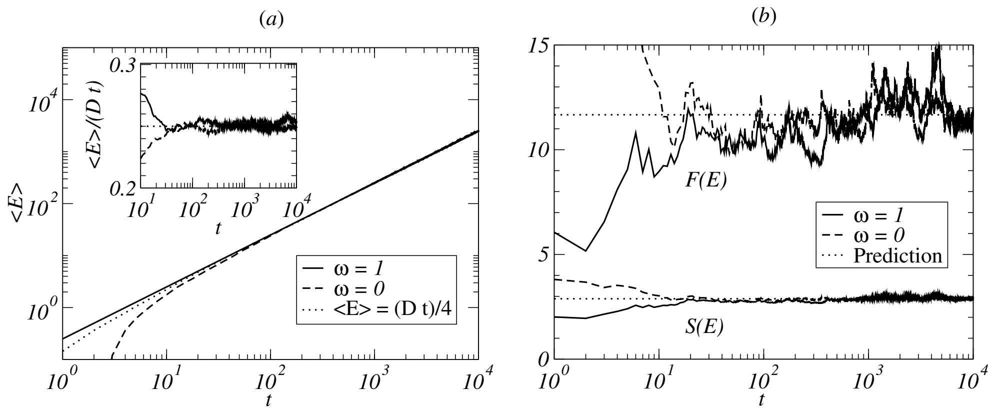

Figure 1. Stochastic pendulum with Gaussian white noise: equations (1), (2), (4) are integrated numerically for $\mathcal{D} = 1$. Ensemble averages are computed over 10^4 realizations. For numerical values of the pulsation $\omega = 1.0$ and 0.0, we plot: (*a*) the average $\langle E\rangle$ and the ratio $\langle E\rangle/(\mathcal{D}t)$ (inset), (*b*) the skewness and flatness factors of E versus time t. The asymptotic behaviour of these observables agrees with equations (9)–(11) (dotted lines in the figures), irrespective of the value of ω.

and also the skewness and flatness factors

$$S(E) = \frac{\langle E^3\rangle}{\langle E^2\rangle^{3/2}} = \frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{1}{2}\right)^{1/2}}{\Gamma\left(\frac{5}{2}\right)^{3/2}} = \frac{5}{\sqrt{3}} \simeq 2.887\ldots \tag{10}$$

$$F(E) = \frac{\langle E^4\rangle}{\langle E^2\rangle^2} = \frac{\Gamma\left(\frac{9}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{2}\right)^2} = \frac{35}{3} \tag{11}$$

where $\Gamma()$ is the Euler Gamma function. We conclude from equation (9) that the average energy of a frictionless pendulum with white parametric noise grows linearly with time. These results, equations (9)–(11) agree with numerical simulations (see figure 1).

2.2. *Scaling analysis for the coloured noise*

When the noise $\xi(t)$ is coloured, it is not possible to write a closed equation for the PDF $P_t(\theta, \Omega)$. However, the scaling behaviour of the dynamical variables can be deduced from a self-consistent reasoning similar to that used in [15, 16]. Suppose *a priori* that we have the following scaling behaviour in the long time limit,

$$\Omega \sim t^{\alpha} \tag{12}$$

where α is an unknown exponent to be determined. We find from equation (2) that $\theta \sim t^{\alpha+1}$, and equation (1) can then be written as

$$\dot{\Omega} \simeq \xi(t) \sin t^{\alpha+1}. \tag{13}$$

(We could have retained the deterministic term $-\omega^2 \sin\theta$ but it would not affect this scaling analysis.) We now take $\xi(t)$ to be a discrete dichotomous noise with correlation time τ and with values ± 1. The previous equation, then, becomes

$$\Omega(t) \sim \sum_{k=1}^{t/\tau} \epsilon_k \int_{(k-1)\tau}^{k\tau} \sin x^{\alpha+1}\mathrm{d}x \qquad \text{with} \quad \epsilon_k = \pm 1. \tag{14}$$

We estimate the last integral by an integration by parts:

$$(\alpha + 1)\int_t^{t+\tau} x^\alpha \frac{\sin x^{\alpha+1}}{x^\alpha}\mathrm{d}x = \left[-\frac{\cos(x^{\alpha+1})}{x^\alpha}\right]_t^{t+\tau} + \mathcal{O}(t^{-\alpha-1}) \tag{15}$$

and obtain

$$\langle\Omega^2\rangle \sim \sum_{k=1}^{t/\tau}\left(\int_{(k-1)\tau}^{k\tau} \sin x^{\alpha+1}\mathrm{d}x\right)^2 \sim \sum_{k=1}^{t/\tau}\frac{1}{(k\tau)^{2\alpha}} \sim t^{1-2\alpha}. \tag{16}$$

This result is compatible with the *a priori* scaling Ansatz (12) only for $\alpha = 1/4$. We thus conclude from this qualitative argument that, in the presence of coloured noise, the energy E of the system defined in equation (3) grows as the square-root of time (as opposed to the linear growth obtained for the white noise). A non-zero correlation time of the noise, however small, modifies the long time scaling behaviour of the system.

In the next section, we develop a method to put this qualitative analysis on a firm basis and derive precise analytic expressions that can be compared quantitatively with numerical results.

3. The averaging method for Ornstein–Uhlenbeck noise

From now on, we consider the random noise ξ to be an Ornstein–Uhlenbeck process, i.e., a Gaussian coloured noise with correlation function given by

$$\langle\xi(t)\xi(t')\rangle = \frac{\mathcal{D}}{2\tau}\,\mathrm{e}^{-|t-t'|/\tau} \tag{17}$$

where τ is the correlation time of the noise. This noise ξ can be generated from the white noise via the Ornstein–Uhlenbeck equation

$$\dot{\xi} = -\frac{1}{\tau}\xi + \frac{1}{\tau}\eta(t) \tag{18}$$

where $\eta(t)$ is a white noise of auto-correlation function $\mathcal{D}\delta(t - t')$. In the stationary limit, $t, t' \gg \tau$, the solution of equation (18) satisfies equation (17). The pendulum with Ornstein–Uhlenbeck noise is thus written as a three-dimensional stochastic dynamical system coupled to a white noise $\eta(t)$:

$$\dot{\Omega} = -\omega^2\sin\theta - \xi\sin\theta \tag{19}$$

$$\dot{\theta} = \Omega \tag{20}$$

$$\dot{\xi} = -\frac{1}{\tau}\xi + \frac{1}{\tau}\eta(t). \tag{21}$$

The Fokker–Planck equation for the three-dimensional PDF $P_t(\theta, \Omega, \xi)$ is given by

$$\frac{\partial P_t}{\partial t} = -\frac{\partial}{\partial\theta}(\Omega P_t) + \frac{\partial}{\partial\Omega}((\omega^2 + \xi)\sin\theta P_t) + \frac{1}{\tau}\frac{\partial}{\partial\xi}(\xi P_t) + \frac{\mathcal{D}}{2\tau^2}\frac{\partial^2 P_t}{\partial\xi^2}. \tag{22}$$

For coloured noise, there is no closed Fokker–Planck equation for the original PDF on the phase space, $P_t(\theta, \Omega)$, and only approximate evolution equations can be written. We shall not use any effective dynamics to derive our results but rather start with the exact three-dimensional Fokker–Planck equation (22) from which we shall integrate out the fast variable.

3.1. Zeroth-order averaging

We show here that the averaging procedure used in section 2.1 for the white noise leads to erroneous results for the coloured noise. Averaging the Fokker–Planck equation (22) over the fast angular variable θ, we find that the marginal PDF $\tilde{P}_t(\Omega, \xi)$ obeys the Ornstein–Uhlenbeck diffusion equation

$$\frac{\partial \tilde{P}_t}{\partial t} = \frac{1}{\tau}\frac{\partial}{\partial \xi}(\xi \tilde{P}_t) + \frac{\mathcal{D}}{2\tau^2}\frac{\partial^2 \tilde{P}_t}{\partial \xi^2} \tag{23}$$

the variable Ω no more appears in this averaged Fokker–Planck equation and the associated stochastic two-dimensional system reads

$$\dot{\Omega} = 0 \qquad \text{and} \qquad \dot{\xi} = -\frac{1}{\tau}\xi + \frac{1}{\tau}\eta(t). \tag{24}$$

In this averaged system, the angular velocity Ω is no more stochastic and, even worse, it is constant in time. By integrating out the fast variable θ without taking into account the correlations between θ and ξ, the dynamical variable Ω has been decoupled from the noise; in other words, the noise itself has been averaged out of the system. In fact, when the typical variation time of θ (i.e. the time during which θ varies by 2π) becomes less than τ, the noise ξ is roughly constant during a period of $\sin\theta$. Thus, if θ and ξ are (wrongly) treated as independent variables, $(\sin\theta)\xi$ is averaged to 0 at the leading order. This problem did not occur in section 2.1 where ξ was a white noise. In that case, the rapid fluctuations of the phase do not wipe out the noise, and $(\sin\theta)\xi$ is averaged to $\xi/\sqrt{2}$ to yield equation (6).

In next sections, we develop an averaging scheme that allows us to eliminate adiabatically the fast variable while retaining the correlation terms. The idea is to define recursively a new set of dynamical variables that embodies the correlations order by order. This will enable us to derive sound asymptotic results for the pendulum with coloured noise. In this scheme, equation (24) appears as a zeroth order approximation, and its correct interpretation is not that Ω is conserved but that its variations are slower than that of normal diffusion.

3.2. First-order averaging

Multiplying both sides of equation (19) by Ω, and using equation (20), we obtain

$$\Omega\dot{\Omega} = -\Omega(\omega^2 \sin\theta + \xi \sin\theta) = -\omega^2\dot{\theta}\sin\theta - \xi(\dot{\theta}\sin\theta). \tag{25}$$

Introducing the energy E of the system defined in equation (3), this equation becomes

$$\frac{\mathrm{dE}}{\mathrm{dt}} = \frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\Omega^2}{2} - \omega^2\cos\theta\right) = -\xi(\dot{\theta}\sin\theta). \tag{26}$$

We now transform the right-hand side by writing it as a total derivative plus a correction term

$$\frac{\mathrm{dE}}{\mathrm{dt}} = \frac{\mathrm{d}}{\mathrm{dt}}(\xi\cos\theta) - \dot{\xi}\cos\theta. \tag{27}$$

Using equation (21), we obtain

$$\frac{\mathrm{d}}{\mathrm{dt}}(E - \xi\cos\theta) = \frac{\cos\theta}{\tau}\xi - \frac{\cos\theta}{\tau}\eta. \tag{28}$$

This leads us to define a new dynamical variable E_1

$$E_1 = E - \xi\cos\theta = \frac{\Omega^2}{2} - (\omega^2 + \xi)\cos\theta \tag{29}$$

and to rewrite the stochastic system (19)–(21) in terms of the set of variables (E_1, θ, ξ)

$$\dot{E}_1 = \frac{\cos\theta}{\tau}\xi - \frac{\cos\theta}{\tau}\eta(t) \tag{30}$$

$$\dot{\theta} = \Omega(E_1, \theta, \xi) = \sqrt{2(E_1 + (\omega^2 + \xi)\cos\theta)} \tag{31}$$

$$\dot{\xi} = -\frac{1}{\tau}\xi + \frac{1}{\tau}\eta(t). \tag{32}$$

The advantage of this system as compared to the previous one (19)–(21) is that the white noise $\eta(t)$ now appears in the equation for the dynamical variable E_1 and this white noise contribution will survive the averaging process. The Fokker–Planck equation for the PDF $P_t(E_1, \theta, \xi)$ associated with equations (30)–(32) reads

$$\frac{\partial P_t}{\partial t} = -\frac{\partial}{\partial E_1}\left(\frac{\cos\theta}{\tau}\xi P_t\right) - \frac{\partial}{\partial\theta}(\Omega(E_1, \theta, \xi)P_t) + \frac{1}{\tau}\frac{\partial}{\partial\xi}(\xi P_t) + \frac{\mathcal{D}}{2\tau^2}\left\{\cos^2\theta\frac{\partial^2 P_t}{\partial E_1^2} - 2\cos\theta\frac{\partial^2 P_t}{\partial E_1\partial\xi} + \frac{\partial^2 P_t}{\partial\xi^2}\right\}. \tag{33}$$

Averaging this equation with respect to θ leads to the following evolution equation for the marginal distribution $\tilde{P}_t(E_1, \xi)$:

$$\frac{\partial\tilde{P}_t}{\partial t} = \frac{1}{\tau}\frac{\partial}{\partial\xi}(\xi\tilde{P}_t) + \frac{\mathcal{D}}{4\tau^2}\frac{\partial^2\tilde{P}_t}{\partial E_1^2} + \frac{\mathcal{D}}{2\tau^2}\frac{\partial^2\tilde{P}_t}{\partial\xi^2}. \tag{34}$$

The variable E_1 now appears in the averaged Fokker–Planck equation, and the associated stochastic two-dimensional system reads

$$\dot{E}_1 = \frac{1}{\sqrt{2}\tau}\eta_1(t) \tag{35}$$

$$\dot{\xi} = -\frac{1}{\tau}\xi + \frac{1}{\tau}\eta(t) \tag{36}$$

where $\eta(t)$ and $\eta_1(t)$ are two independent white noises of amplitude $\mathcal{D}$. This effective dynamics is exactly solvable (the variables E_1 and ξ are decoupled, thanks to the absence of the second-order cross-derivative term in equation (34)). If we compare this system with the one obtained by naive averaging (24), we observe that the dynamical variable E_1 (and therefore the energy of the system) is no longer constant, rather it grows as the square-root of time. When $t \to \infty$, E_1 becomes identical to the energy of the system (up to terms that remain finite). We therefore determine the long time statistics of the energy from equation (35) by identifying E_1 to E and by imposing the physical condition $E \geqslant 0$. The energy is thus a Wiener process on a half-line and its PDF is given by

$$P_t(E) = \frac{2\tau}{\sqrt{\pi\mathcal{D}t}}\exp\left(-\frac{\tau^2E^2}{\mathcal{D}t}\right) \qquad \text{with} \qquad E \geqslant 0. \tag{37}$$

From this expression we calculate the first two moments of the energy

$$\langle E\rangle = \frac{1}{\sqrt{\pi}}\left(\frac{\mathcal{D}t}{\tau^2}\right)^{\frac{1}{2}} \simeq 0.564\left(\frac{\mathcal{D}t}{\tau^2}\right)^{\frac{1}{2}} \tag{38}$$

$$\langle E^2\rangle = \frac{1}{2}\frac{\mathcal{D}t}{\tau^2}. \tag{39}$$

These expressions provide scaling relations between the averages, the time t and the dimensional parameters of the problem, $\mathcal{D}$ and τ. We have verified numerically that these

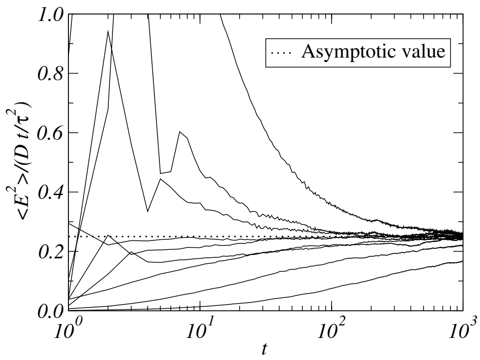

Figure 2. Stochastic pendulum with Ornstein–Uhlenbeck coloured noise: equations (19)–(21) are integrated numerically for $\omega = 1.0$ and $(\mathcal{D}, \tau) = (1, 0.1), (1, 1), (1, 10), (10, 0.1), (10, 1), (10, 10), (100, 0.1), (100, 1), (100, 10)$. Ensemble averages are computed over 10^4 realizations. We plot the ratio $\langle E^2\rangle/(\mathcal{D}t/\tau^2)$ versus time t. The dotted line in the figure corresponds to the asymptotic value 0.25 predicted by second-order averaging (equation (55)). Note that the convergence is slower for smaller values of the correlation time τ.

scalings are correct (see figure 2). However, the prefactors that appear in equations (38) and (39) are pure numbers and do not agree with the results of our numerical simulations. We conclude that equation (38) is exact at the leading order as it gives the correct asymptotic scaling for the energy, $E \propto t^{1/2}$ but fails to provide the prefactors. The reason is that some correlations between θ and ξ have still been neglected in the averaging procedure. Carrying out the calculations to the next higher order will enable us to derive the correct expressions for the prefactors.

3.3. Second-order averaging

More precise results can indeed be derived by applying recursively the procedure described above. In the Langevin equation (30) for E_1, we perform one more 'integration by parts' and obtain

$$\dot{E}_1 = \frac{\xi\cos\theta}{\tau}\frac{\dot\theta}{\Omega} - \frac{\cos\theta}{\tau}\eta(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\xi\sin\theta}{\tau\Omega}\right) - \frac{\sin\theta}{\tau\Omega}\dot\xi + \frac{\dot\Omega}{\Omega^2}\frac{\sin\theta}{\tau}\xi - \frac{\cos\theta}{\tau}\eta(t). \tag{40}$$

This leads us to introduce a new variable E_2 defined as

$$E_2 = E_1 - \frac{\xi\sin\theta}{\tau\Omega} = \frac{\Omega^2}{2} - (\omega^2+\xi)\cos\theta - \frac{\xi\sin\theta}{\tau\Omega}. \tag{41}$$

Using equations (19) and (21), equation (40) becomes

$$\dot{E}_2 = -\frac{\xi\sin\theta}{\tau^2\Omega} - \frac{\xi\sin^2\theta}{\tau\Omega^2}(\omega^2+\xi) - \left(\cos\theta + \frac{\sin\theta}{\tau\Omega}\right)\frac{\eta(t)}{\tau}. \tag{42}$$

In this equation we must express the variable Ω in terms of E_2, θ and ξ. Inverting relation (41), we deduce that

$$\Omega = (2E_2)^{\frac{1}{2}} + (\omega^2+\xi)\frac{\cos\theta}{(2E_2)^{\frac{1}{2}}} + \frac{\xi\sin\theta}{\tau(2E_2)} + \mathcal{O}\left(\frac{1}{E_2^{\frac{3}{2}}}\right) \tag{43}$$

where we have retained terms up to the order $1/E_2$. From equation (42), we deduce the Langevin equation for E_2

$$\dot{E}_2 = J_E(E_2, \theta, \xi) + D_E(E_2, \theta, \xi)\frac{\eta(t)}{\tau} + \mathcal{O}\left(\frac{1}{E_2^{\frac{3}{2}}}\right) \tag{44}$$

where we have defined

$$J_E(E_2, \theta, \xi) = -\frac{\xi \sin\theta}{\tau^2(2E_2)^{\frac{1}{2}}} - \frac{\xi \sin^2\theta}{\tau(2E_2)}(\omega^2 + \xi) \tag{45}$$

$$D_E(E_2, \theta, \xi) = -\left(\cos\theta + \frac{\sin\theta}{\tau(2E_2)^{\frac{1}{2}}}\right). \tag{46}$$

This equation, combined with equations (20) and (21), defines a three-dimensional stochastic system for the variables (E_2, θ, ξ). The Fokker–Planck equation for the PDF $P_t(E_2, \theta, \xi)$ is

$$\frac{\partial P_t}{\partial t} = -\frac{\partial}{\partial E_2}(J_E P_t) - \frac{\partial}{\partial \theta}(\Omega(E_2, \theta, \xi)P_t) + \frac{1}{\tau}\frac{\partial}{\partial \xi}(\xi P_t)$$
$$+ \frac{\mathcal{D}}{2\tau^2}\left\{\frac{\partial}{\partial E_2}D_E\frac{\partial}{\partial E_2}(D_E P_t) + \frac{\partial^2}{\partial\xi\partial E_2}(D_E P_t) + \frac{\partial}{\partial E_2}D_E\frac{\partial P_t}{\partial \xi} + \frac{\partial^2 P_t}{\partial \xi^2}\right\}. \tag{47}$$

We now integrate out the fast angular variable θ from equation (47), retaining only the leading term in the average of the expression $\frac{\partial}{\partial E_2}D_E\frac{\partial}{\partial E_2}(D_E P_t)$ (recalling that $\partial/\partial E_2$ scales as E_2^{-1}, the contribution of the subdominant terms is of the order of $E_2^{-5/2}$ and is negligible in the long time limit). We thus obtain the following evolution equation for the marginal distribution $\tilde{P}_t(E_2, \xi)$:

$$\frac{\partial \tilde{P}_t}{\partial t} = \frac{\partial}{\partial E_2}\left(\frac{\omega^2\xi + \xi^2}{4\tau E_2}\tilde{P}_t\right) + \frac{1}{\tau}\frac{\partial}{\partial\xi}(\xi\tilde{P}_t) + \frac{\mathcal{D}}{2\tau^2}\left\{\frac{1}{2}\frac{\partial^2\tilde{P}_t}{\partial E_2^2} + \frac{\partial^2\tilde{P}_t}{\partial\xi^2}\right\}. \tag{48}$$

Although the cross-derivative terms between E_2 and ξ vanish, the two variables are coupled through the drift term. From equation (48) we derive an effective, two-dimensional, stochastic system in E_2 and ξ:

$$\dot{E}_2 = -\frac{\omega^2\xi + \xi^2}{4\tau E_2} + \frac{1}{\sqrt{2}\tau}\eta_2(t) \tag{49}$$

$$\dot{\xi} = -\frac{1}{\tau}\xi + \frac{1}{\tau}\eta(t) \tag{50}$$

$\eta(t)$ and $\eta_2(t)$ being two independent white noises of amplitude $\mathcal{D}$. If we compare this system deduced by a second-order averaging with the one obtained at first-order (equations (35), (36)), we observe that the equation for the dynamical variable E_2 contains, besides a noise term, an effective potential that scales like $1/E_2$ and that involves the other variable ξ. This effective potential diverges at $E_2 = 0$ and constrains the energy to stay positive: we no longer need to impose this physical condition arbitrarily.

3.4. Analytical results

In the coupled system (49), (50), the fast angular variable has been eliminated and the dimensionality of the original problem has been reduced by one. However, equation (49) is nonlinear and is not exactly solvable. Nevertheless, the long time behaviour of E_2 can be deduced from the following reasoning. Recalling that ξ^2 has a finite mean value, equal to $\mathcal{D}/2\tau$, we rewrite the evolution equation (49) of E_2 as

$$\dot{E}_2 = -\frac{\mathcal{D}}{8\tau^2 E_2} + \frac{1}{\sqrt{2}\tau}\eta_2(t) - \frac{1}{4\tau E_2}(\omega^2\xi + \xi^2 - \langle\xi^2\rangle). \tag{51}$$

This Langevin equation contains two independent noise contributions: a white noise $\eta_2(t)$, and a (non-Gaussian) coloured noise, $(\omega^2\xi + \xi^2 - \langle\xi^2\rangle)$, of zero mean value and of finite variance. This coloured noise is multiplied by a prefactor proportional to $1/E_2$ and, because E_2 goes to infinity with time, it becomes negligible in the large time limit in comparison with the white noise term. Thus, equation (51) reduces asymptotically to

$$\dot{E}_2 = -\frac{\mathcal{D}}{8\tau^2 E_2} + \frac{1}{\sqrt{2}\tau}\eta_2(t). \tag{52}$$

The variables E_2 and ξ are decoupled at large times: the effective problem is thus one-dimensional. In the long time limit, the variable E_2 is identical to the energy E up to finite terms. We thus obtain the asymptotic PDF of the pendulum's energy by explicitly solving the Fokker–Planck associated with equation (52),

$$\tilde{P}_t(E) = \frac{2\sqrt{\tau}}{\Gamma\left(\frac{1}{4}\right)(\mathcal{D}t)^{1/4}} E^{-\frac{1}{2}} \exp\left(-\frac{\tau^2 E^2}{\mathcal{D}t}\right). \tag{53}$$

Hence, E is not a Wiener process on a half line: its PDF is not a simple Gaussian. From equation (53), we calculate the first two moments of the energy

$$\langle E\rangle = \frac{\sqrt{2}\pi}{\Gamma\left(\frac{1}{4}\right)^2}\left(\frac{\mathcal{D}t}{\tau^2}\right)^{1/2} \simeq 0.338\left(\frac{\mathcal{D}t}{\tau^2}\right)^{1/2} \tag{54}$$

$$\langle E^2\rangle = \frac{1}{4}\frac{\mathcal{D}t}{\tau^2}. \tag{55}$$

Besides the skewness and the flatness factors are

$$S(E) = \frac{\Gamma\left(\frac{7}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{5}{4}\right)^{3/2}} = \frac{6\sqrt{2}\pi}{\Gamma\left(\frac{1}{4}\right)^2} \simeq 2.028\ldots \tag{56}$$

$$F(E) = \frac{\Gamma\left(\frac{9}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{5}{4}\right)^2} = 5. \tag{57}$$

The functional dependence of the moments on time t and on the parameters $\mathcal{D}$ and τ is the same as that obtained in section 3.2. But the prefactors, which are absolute numbers, are different (compare equations (38), (39) with equations (54), (55)). Numerical simulations of the dynamical equations (19)–(21), shown in figures 2 and 3, agree quantitatively with the predictions of equations (54)–(57). In particular, we note that the asymptotic PDF $\tilde{P}_t(E)$, and therefore the moments of the energy, do not depend on the value of the mean frequency ω (see figure 3).

Our averaging technique thus provides sound asymptotic results for the energy of the stochastic pendulum: this technique not only yields the correct scalings but also leads to the analytical formulae for the large time behaviour of the energy. This averaging method could be carried over to the third order to calculate the subdominant corrections to the PDF of the energy. However, we shall not pursue this course any further: the calculations become very unwieldy and the agreement between the analytical results and the numerical computations is already very satisfactory at the second order.

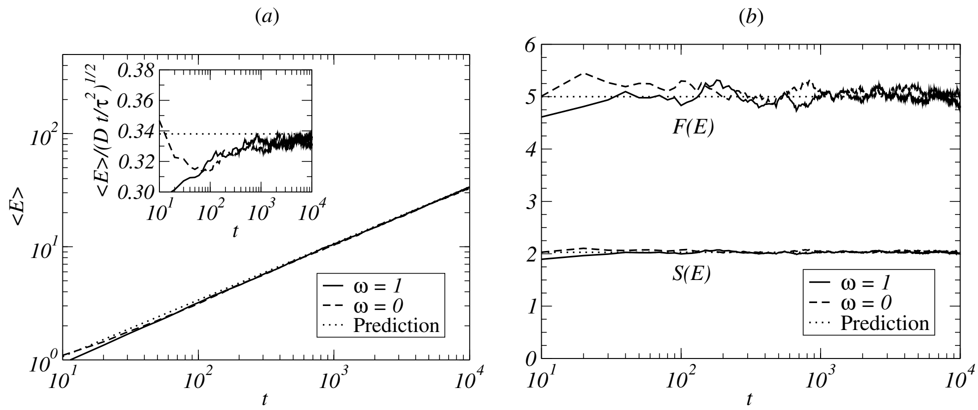

Figure 3. Stochastic pendulum with Ornstein–Uhlenbeck coloured noise: equations (19)–(21) are integrated numerically for $\mathcal{D} = 1, \tau = 1$. Ensemble averages are computed for over 10^4 realizations. For numerical values of the pulsation $\omega = 1.0$ and 0.0, we plot: (*a*) the average $\langle E\rangle$ and the ratio $\langle E\rangle/(\mathcal{D}t/\tau^2)^{1/2}$ (inset), (*b*) the skewness and flatness factors of *E* versus time *t*. The asymptotic behaviour of all observables presented agrees with equations (54)–(57) (dotted lines in the figures), irrespective of the value of ω.

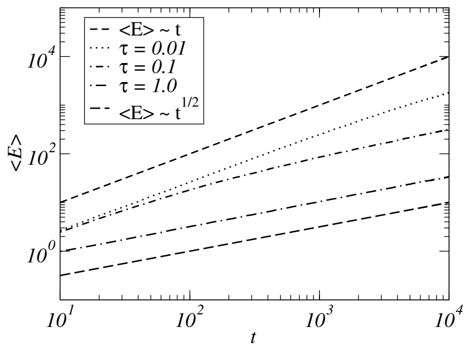

Figure 4. Stochastic pendulum with Ornstein–Uhlenbeck coloured noise: equations (19)–(21) are integrated numerically for $\omega = 0.0, \mathcal{D} = 1.0$ and $\tau = 0.01, 0.1$ and 1. Ensemble averages are computed over 10^4 realizations. We plot the average energy versus time. The crossover between the white noise behaviour ($\langle E\rangle \sim t$) and the coloured noise behaviour ($\langle E\rangle \sim t^{1/2}$) occurs later for smaller correlation time τ.

We emphasize that the white noise and the coloured noise cases fall into two distinct universality classes because the long time scaling exponents are different. The coloured noise scaling will always be observed after a sufficiently long time provided that the correlation time τ is non-zero. However, the effect of this correlation time appears only when the period *T* of the pendulum is less than τ. This period, which is proportional to Ω^{-1}, decreases with time. At short times, the period *T* is much greater than τ and white noise scalings are observed. At large times, $T \ll \tau$ and coloured noise scalings are satisfied. The crossover time t_c is reached when $T \sim \Omega^{-1} \sim \tau$; using equation (9), which is valid for $t < t_c$, we obtain

$$t_c \sim (\mathcal{D}\tau^2)^{-1}. \tag{58}$$

Hence, when the correlation time τ becomes vanishingly small, the crossover time diverges to infinity and the coloured noise regime is not reached (the simulation times needed to observe the coloured noise scalings become increasingly long). In figure 4, we plot the behaviour of the mean energy versus time for $\mathcal{D} = 1.0$ and $\tau = 0.01, 0.1$ and 1. For $\tau = 1$, the coloured noise scaling regime is obtained from the very beginning and the curve for $\langle E \rangle$ has a slope $1/2$ in the log–log scale. For $\tau = 0.1$, at short times $\langle E \rangle \propto t$ whereas at long times $\langle E \rangle \propto t^{1/2}$. The crossover is observed around $t_c \sim 100 = \tau^{-2}$. For $\tau = 0.01$, the curve for $\langle E \rangle$ has a slope 1 and the coloured noise regime is not reached in this simulation ($t_c \sim 10^4$).

The averaging method provides analytical results for $t \gg t_c$ and $t \ll t_c$. The intermediate time regime is described by a crossover function of the scaled variable t/t_c [15], which cannot be analysed by our technique. In the next section, we compare our method with two well-known approximations based on effective coloured noise Fokker–Planck equations. One of these approaches will enable us to derive approximate formulae for t_c and for the crossover function.

4. Comparison with other approximation schemes

The main difficulty for the study of a Langevin equation with coloured noise stems from its non-Markovian character [1]: there exists no closed Fokker–Planck equation that describes the evolution of the PDF of the dynamical variables. The process can be embedded into a Markov system if the coloured noise itself is treated as a variable. However, this mathematical trick increases the dimension of the problem by one and the noise has to be integrated out in the end. Exact calculations can be carried out with this method only in the case of linear problems.

A coloured noise master equation can be rigorously derived (e.g., with the help of functional methods) but it involves correlations between the dynamical variables and the noise [23, 24]. The equation of motion for these correlations involves higher order correlations, and so on. Since this hierarchy must be stopped at some stage, this question is a genuine closure problem. Many different approaches have been devoted to derive effective Fokker–Planck equations, such as short correlation time expansions [17, 18, 25, 26], unified coloured noise approximation [27], projection methods [28] and self-consistent decoupling Ansätz [29] (for a general review see [30]).

In section 3, to derive the long time behaviour of the stochastic pendulum with the coloured noise, we did not use any closure approximation but started from the exact Fokker–Planck equation for the system, treating the noise as an auxiliary variable. Thus, we did not make any hypothesis on τ and our analytical formulae are valid for any value of the correlation time (for t larger than the crossover time given in equation (58)). In this section, we compare our results with those that can be derived from some well-known approximation schemes.

4.1. Small correlation time expansion

An approximate evolution equation for the PDF of a Langevin equation with coloured noise can be derived in the case of short correlation times (i.e., in the white noise limit) by expanding the coloured noise master equation around the Markovian point [26, 30]. This procedure leads to a Fokker–Planck-type equation, with effective drift and diffusion coefficients. Applying to our system (19)–(21) the small τ expansion derived in [26] for arbitrary stochastic equations with coloured noise, we obtain, at first order in τ, the effective Fokker–Planck equation for $P_t(\theta, \Omega)$

$$\frac{\partial P_t}{\partial t} = -\frac{\partial}{\partial \theta}(\Omega P_t) - \frac{3\mathcal{D}\tau}{2}\sin\theta\cos\theta\frac{\partial P_t}{\partial \Omega} + \frac{\mathcal{D}(1-\tau\Omega)}{2}\sin^2\theta\frac{\partial^2 P_t}{\partial \Omega^2} + \frac{\mathcal{D}\tau}{2}\frac{\partial^2}{\partial\theta\partial\Omega}(\sin^2\theta P_t). \tag{59}$$

To simplify the discussion, we have taken the mean frequency ω equal to zero. After integrating out the rapid variations of θ an averaged Fokker–Planck equation is obtained which is similar to equation (6), but with an effective diffusion given by $\mathcal{D}_{\text{eff}} = \mathcal{D}(1 - \tau\Omega)$ (because $\mathcal{D}_{\text{eff}}$ becomes negative for large values of Ω, equation (59) is valid only over the restricted region of positive diffusion). From dimensional analysis, we observe that such an effective Fokker–Planck equation leads to a normal diffusive behaviour, $\Omega \sim t^{1/2}$ and $E \sim t$, and therefore cannot account for the results we have obtained.

4.2. *Best effective Fokker–Planck equation*

We now turn to another small τ approximation, which intends to improve the first-order effective Fokker–Planck equation (59) by summing contributions of the type $\mathcal{D}\tau^n$ (where n is an integer larger than 1). The resulting equation has been christened 'best Fokker–Planck equation' (BFPE) by its proponents [17, 18, 31]. Although this approach is not free from drawbacks and is known to lead in some cases to unphysical results [32], we show that for the system studied in this work, the BFPE leads to results that agree with ours.

An approximate evolution equation for the PDF $P_t(\Omega, \theta)$ is given by the second-order cumulant expansion [1] of the (stochastic) Liouville equation associated with equations (19) and (20):

$$\frac{\partial P_t}{\partial t} = \mathbf{L}_0 P_t + \int_0^t \mathrm{d}x \, \langle \mathbf{L}_1(t) \exp(\mathbf{L}_0 x) \mathbf{L}_1(t-x) \exp(-\mathbf{L}_0 x) \rangle P_t \tag{60}$$

where the differential operators are defined as

$$\mathbf{L}_0 P_t = -\frac{\partial}{\partial\theta}(\Omega P_t) \tag{61}$$

$$\mathbf{L}_1(t) P_t = -\frac{\partial}{\partial\Omega}(\xi(t) \sin\theta \, P_t). \tag{62}$$

(Here again we take $\omega = 0$.) In the appendix, we evaluate the right-hand side of equation (60) and derive, in the limit $t \to \infty$, the following BFPE for the classical pendulum with coloured multiplicative noise

$$\begin{aligned}\frac{\partial P_t}{\partial t} = -\frac{\partial}{\partial\theta}(\Omega P_t) + \frac{\mathcal{D}}{2}\frac{\partial^2}{\partial\Omega^2}\left(\frac{\sin^2\theta - \tau\Omega\sin\theta\cos\theta}{1 + (\tau\Omega)^2} P_t\right) \\ + \frac{\mathcal{D}\tau}{2}\frac{\partial}{\partial\Omega}\sin\theta\frac{\partial}{\partial\theta}\left(\frac{(1 - (\tau\Omega)^2)\sin\theta - 2\tau\Omega\cos\theta}{(1 + (\tau\Omega)^2)^2} P_t\right).\end{aligned} \tag{63}$$

Integrating out the fast variable θ, we obtain an averaged BFPE for $\tilde{P}_t(\Omega)$, the probability distribution of the slow variable,

$$\frac{\partial \tilde{P}_t}{\partial t} = \frac{\mathcal{D}}{4}\frac{\partial}{\partial\Omega}\left(\frac{1}{1 + (\tau\Omega)^2}\frac{\partial \tilde{P}_t}{\partial\Omega}\right). \tag{64}$$

For $\tau = 0$, this equation is identical to the averaged white noise Fokker–Planck equation (6). For a non-zero correlation time, this equation predicts correctly that Ω grows as $t^{1/4}$: this is straightforward from scaling. The crossover between the white and the coloured noises is observed for $\Omega \sim 1/\tau$ and this is consistent with the discussion that led to equation (58). Although $\tilde{P}_t(\Omega)$ is not explicitly calculable, equation (64) implies the following identity:

$$\frac{\tau^2}{4}\langle\Omega^4\rangle + \frac{1}{2}\langle\Omega^2\rangle = \frac{\mathcal{D}}{4}t. \tag{65}$$

When t is small, the quadratic term dominates over the quartic term, and we recover $\langle\Omega^2\rangle = 2\langle E\rangle \simeq \frac{\mathcal{D}}{2}t$, in agreement with equation (9). When t is large, the quartic term is dominant and one deduces that

$$\langle E^2\rangle = \frac{1}{4}\langle\Omega^4\rangle \simeq \frac{1}{4}\frac{\mathcal{D}}{\tau^2}t. \tag{66}$$

This result is identical to our equation (55), which was validated by numerical results (see figure 2). Identity (65) can also be used to derive an approximate scaling function for the mean energy. Let us define the flatness ϕ of $\tilde{P}_t(\Omega)$ as

$$\langle\Omega^4\rangle = \phi\langle\Omega^2\rangle^2. \tag{67}$$

Rigorously speaking ϕ is a function of time, but it remains a number of order 1. For simplicity, let us assume that ϕ is constant. Substituting equation (67) into equation (65), and solving for $\langle\Omega^2\rangle$, we obtain

$$\langle E\rangle = \frac{1}{2}\langle\Omega^2\rangle = \frac{\sqrt{\mathcal{D}\tau^2\phi t + 1} - 1}{2\tau^2\phi}. \tag{68}$$

This scaling function explicitly describes the evolution of the energy of the oscillator as a function of time. It contains, in particular, the linear behaviour at short time and the $t^{1/2}$ growth at large time. The crossover between these two scaling regimes occurs when $\mathcal{D}\tau^2 t \sim 1$, i.e., precisely at the crossover time given in equation (58).

The BFPE approach has thus allowed us to derive short and long time scalings and to understand qualitatively the evolution of the system at the intermediate times. The BFPE is derived from a second-order cumulant expansion in which the higher order terms, which are not Fokker–Planck-like, are discarded. This approximation has been criticized [32] because the neglected higher-derivative terms can be of the same order as the terms that have been retained in the BFPE However, for the stochastic pendulum, these neglected terms do not qualitatively change the large time behaviour of the dynamical variables. In contrast, the averaging technique that we have generalized here is not based on any *a priori* expansion and provides reliable results at least for the stochastic system studied in this work. Of course, the approach advocated here has been developed for one particular problem and does not have the generality and versatility of the effective Fokker–Planck methods. Nevertheless, we strongly believe that the recursive averaging scheme developed here for the stochastic pendulum can be extended to other nonlinear one-dimensional systems subject to multiplicative or additive coloured noise [33].

5. Conclusion

A nonlinear pendulum subject to parametric noise undergoes a noise-induced diffusion in the phase space. The characteristics of this motion depend on the nature of the randomness: when the noise is white the energy of the pendulum grows linearly with time, whereas it varies only as the square root of time when the noise is coloured. This change of behaviour is due to the destructive interference between the displacement of the pendulum and the noise term: the effect of the coloured noise is partially averaged out by fast angular variations as soon as the period of the pendulum becomes smaller than the correlation time τ of the noise. We have carried out an analytical study of this model by defining recursively new coordinates in the phase space and averaging out the fast angular variable. At zeroth order, the only information obtained is that the motion is subdiffusive; at first order, this procedure provides the correct scalings; at second order, a quantitative agreement with numerical simulations is reached.

We emphasize that our method is different from the usual approximations that involve the effective Fokker–Planck equations in which the coloured noise does not appear as an auxiliary variable. Our averaging procedure integrates out the fast dynamical variable and leads to an effective stochastic dynamics for the slow variables with coloured noise. Whereas the usual effective Fokker–Planck equations are valid only for small noise and for short correlation times, we do not make any hypothesis on the amplitude of the noise or on the correlation time τ. However, the asymptotic subdiffusive regime is reached earlier for larger values of τ. Our results agree with those derived after averaging the 'best Fokker–Planck equation' (BFPE) approximation (which is obtained from a summation of a truncated cumulant expansion of the Liouville equation). This approximation is also useful to draw a qualitative physical picture of the system and allows us to calculate approximate crossover functions between short time (white noise) and long time (coloured noise) regimes.

The recursive averaging scheme that we have used here for the stochastic pendulum can be extended to other systems coupled with coloured noise. In particular, we believe that thanks to this method a precise mathematical analysis of the long time behaviour of a nonlinear oscillator subject to multiplicative or additive coloured noise can be carried out [33].

Acknowledgments

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Appendix A. Derivation of equation (63)

In this appendix, we derive the BFPE (equation (63)) following the procedure of [17, 18]. We evaluate the right-hand side of equation (60) by applying the following operator formula,

$$\exp(A)B\exp(-A) = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \cdots \tag{A.1}$$

with $A = \mathbf{L}_0$ and $B = \mathbf{L}_1(t - x)$. We thus have to calculate some commutators of the two operators $\mathbf{L}_0$ and $\mathbf{L}_1(t - x)$ defined in equations (61) and (62). By induction, we derive the following expression for the nth commutator,

$$T_n = [\mathbf{L}_0, [\ldots, [\mathbf{L}_0, [\mathbf{L}_0, \mathbf{L}_1(t - x)]]\ldots] = \xi(t - x)\left(\frac{\partial}{\partial\Omega}H_1^{(n)}(\Omega, \theta) + \frac{\partial}{\partial\theta}H_2^{(n)}(\Omega, \theta)\right) \tag{A.2}$$

where the functions $H_1^{(n)}$ and $H_2^{(n)}$ satisfy the recursion relations

$$H_1^{(n)} = -\Omega\frac{\partial H_1^{(n-1)}}{\partial\theta} \tag{A.3}$$

$$H_2^{(n)} = H_1^{(n-1)} - \Omega\frac{\partial H_2^{(n-1)}}{\partial\theta}. \tag{A.4}$$

The first few terms can be calculated explicitly and we obtain

$$H_1^{(0)} = -\sin\theta \quad \text{and} \quad H_2^{(0)} = 0 \tag{A.5}$$

$$H_1^{(1)} = \Omega\cos\theta \quad \text{and} \quad H_2^{(1)} = -\sin\theta \tag{A.6}$$

$$H_1^{(2)} = \Omega^2\sin\theta \quad \text{and} \quad H_2^{(2)} = 2\Omega\cos\theta. \tag{A.7}$$

The general solution for recursion (A.4) is readily found:

$$H_1^{(n)} = (-1)^{n-1}\Omega^n \sin\left(\theta + n\frac{\pi}{2}\right) \tag{A.8}$$

$$H_2^{(n)} = nH_1^{(n-1)} = (-1)^{n-1} n\Omega^{n-1} \cos\left(\theta + n\frac{\pi}{2}\right). \tag{A.9}$$

From equations (A.1) and (A.2), we deduce the following identity:

$$\langle \mathbf{L}_1(t)\exp(\mathbf{L}_0 x)\mathbf{L}_1(t-x)\exp(-\mathbf{L}_0 x)\rangle = \sum_{n=0}^{\infty}\frac{x^n}{n!}\langle \mathbf{L}_1(t)T_n\rangle$$
$$= -\frac{\partial}{\partial\Omega}\sin\theta\sum_{n=0}^{\infty}\frac{x^n}{n!}\langle\xi(t)\xi(t-x)\rangle\left(\frac{\partial}{\partial\Omega}H_1^{(n)}(\Omega,\theta) + \frac{\partial}{\partial\theta}H_2^{(n)}(\Omega,\theta)\right). \tag{A.10}$$

Substituting into this equation the expressions (A.8) and (A.9) for $H_1^{(n)}$ and $H_2^{(n)}$, and the autocorrelation function (17) of the Ornstein–Uhlenbeck noise, we find that the right-hand side of equation (60) is given by

$$\int_0^t \mathrm{d}x\,\langle \mathbf{L}_1(t)\exp(\mathbf{L}_0 x)\mathbf{L}_1(t-x)\exp(-\mathbf{L}_0 x)\rangle$$
$$= -\frac{\mathcal{D}}{2\tau}\frac{\partial}{\partial\Omega}\sin\theta\left(\frac{\partial}{\partial\Omega}\mathcal{H}_1(\Omega,\theta,t) + \frac{\partial}{\partial\theta}\mathcal{H}_2(\Omega,\theta,t)\right) \tag{A.11}$$

where we have defined

$$\mathcal{H}_1(\Omega,\theta,t) = \sum_{n=0}^{\infty}\frac{\int_0^t \mathrm{d}x\,x^n\,\mathrm{e}^{-x/\tau}}{n!}H_1^{(n)} = \sum_{n=0}^{\infty}\frac{\int_0^{t/\tau}\mathrm{d}x\,x^n\,\mathrm{e}^{-x}}{n!}(-1)^{n-1}\tau^{n+1}\Omega^n\sin\left(\theta + n\frac{\pi}{2}\right) \tag{A.12}$$

$$\mathcal{H}_2(\Omega,\theta,t) = \sum_{n=0}^{\infty}\frac{\int_0^t \mathrm{d}x\,x^n\,\mathrm{e}^{-x/\tau}}{n!}H_2^{(n)} = \sum_{n=0}^{\infty}\frac{\int_0^{t/\tau}\mathrm{d}x\,x^n\,\mathrm{e}^{-x}}{n!}(-1)^{n-1}\tau^{n+1}n\Omega^{n-1}\cos\left(\theta + n\frac{\pi}{2}\right). \tag{A.13}$$

In the limit $t \to \infty$, the integral $\int_0^{t/\tau}\mathrm{d}x\,x^n\,\mathrm{e}^{-x}$ converges to $n!$, and the series defining $\mathcal{H}_1$ and $\mathcal{H}_2$ can be calculated in a closed form. We finally obtain

$$\mathcal{H}_1(\Omega,\theta,\infty) = -\tau\frac{\sin\theta - \tau\Omega\cos\theta}{1+(\tau\Omega)^2} \tag{A.14}$$

$$\mathcal{H}_2(\Omega,\theta,\infty) = -\tau^2\frac{(1-(\tau\Omega)^2)\sin\theta - 2\tau\Omega\cos\theta}{(1+(\tau\Omega)^2)^2}. \tag{A.15}$$

This completes the proof of equation (63).

References

[1] Van Kampen N G 1992 *Stochastic Processes in Physics and Chemistry* (Amsterdam: North-Holland)

[2] Horsthemke H and Lefever R 1984 *Noise Induced Transitions* (Berlin: Springer)

[3] Anishchenko V S, Astakhov V V, Neiman A B, Vadivasova T E and Schimansky-Geier L 2002 *Nonlinear Dynamics of Chaotic and Stochastic Systems* (Berlin: Springer)

[4] Van den Broeck C, Parrondo J M R and Toral R 1994 *Phys. Rev. Lett.* **73** 3395

[5] San Miguel M and Toral R 1997 *Instabilities and Nonequilibrium Structures VI* ed E Tirapegui and W Zeller (Dordrecht: Kluwer)
[6] Gammaitoni L, Hänggi P and Marchesoni F 1998 *Rev. Mod. Phys.* **70** 223
[7] Ajdari A and Prost J 1992 *C. R. Acad. Sci. Paris* II **315** 1635
[8] Reimann P 2002 *Phys. Rep.* **361** 57
[9] Gardiner C W 1994 *Handbook of Stochastic Methods* (Berlin: Springer)
[10] Landa P S and McClintock P V E 2000 *Phys. Rep.* **323** 1
[11] Stratonovich R L 1963 *Topics on the Theory of Random Noise* vol 1 (New York: Gordon and Breach)
Stratonovich R L 1967 *Topics on the Theory of Random Noise* vol 2 (New York: Gordon and Breach)
[12] Berthet R, Residori S, Roman B and Fauve S 2002 *Phys. Rev. Lett.* **33** 557
[13] Muñoz M A 2003 *Preprint* cond-mat/0303650
[14] Mallick K and Marcq P 2002 *Phys. Rev.* E **66** 041113
[15] Mallick K and Marcq P 2003 *Eur. Phys. J.* B **31** 553
[16] Mallick K and Marcq P 2003 *Physica* A **325** 213
[17] Lindenberg K and West B J 1983 *Physica* A **119** 485
[18] Lindenberg K and West B J 1984 *Physica* A **128** 25
[19] Bourret R C, Frisch U and Pouquet A 1973 *Physica* **65** 303
[20] Van Kampen N G 1976 *Phys. Rep.* **24** 171
[21] Lindenberg K, Seshadri V and West B J 1981 *Physica* A **105** 445
[22] Lindenberg K, Seshadri V and West B J 1980 *Phys. Rev.* A **22** 2171
[23] Hänggi P 1985 *Stochastic Processes Applied to Physics* ed L Pesquera and M A Rodriguez (Singapore: World Scientific)
[24] Hänggi P 1989 *Noise in Dynamical Systems* vol 1 ed F Moss and P V E Mc Clintock (Cambridge: Cambridge University Press)
[25] San Miguel M and Sancho J M 1980 *Phys. Lett.* A **76** 97
[26] Ramirez-Piscina L and Sancho J M 1988 *Phys. Rev.* A **37** 4469
[27] H'walisz L, Jung P, Hänggi P, Talkner P and Schimansky-Geier L 1989 *Z. Phys.* B **77** 471
[28] Fox R F 1986 *Phys. Rev.* A **33** 467
[29] Fronzoni L, Grigolini P, Hänggi P, Moss F, Mannella R and Mc Clintock P V E 1986 *Phys. Rev.* A **33** 3320
[30] Hänggi P and Jung P 1995 *Adv. Chem. Phys.* **89** 239
[31] Peacock-Lopez E, de la Rubia F J, Lindenberg K and West B J 1989 *Phys. Lett.* A **136** 96
[32] Hänggi P, Marchesoni F and Grigolini P 1984 *Z. Phys.* B **56** 333
[33] Mallick K and Marcq P in preparation